Note di Matematica **27,** n. 2, 2007, 195–213.

On Global Equivalence and Entropy in Simple Bodies

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Received: 13/09/2006; accepted: 12/10/2006.

Abstract. In a well known paper W. A. Day pointed out that, for bodies with fading memory, entropy can be considered as a quantity that is not well defined because of its non uniqueness. That remark has been considered as a datum in the criticism of foundations of continuum thermodynamics. The present paper points out that the indetermination of entropy plays a role also in global equivalence for bodies with fading memory.

Keywords: Global Equivalence, Entropy functionals, Simple bodies, Continuum thermodynamics, Entropy indetermination, Entropy equivalent heat fluxes

MSC 2000 classification: primary 74A20

1 Introduction

1.1 Premise

In the mind of people *entropy* is probably the most fascinating physical quantity related to a heat-conducting continuous body.

On the other hand, it is also the most obscure quantity: it does not enter the formulation of the initial and boundary-value problem.

A. Day, in his paper [1], noticed that for bodies with fading memory, entropy can be considered as a quantity that is not well (*i.e.* uniquely) defined.

More in detail, in connection with a rigid heat-conducting body with fading memory, Day showed that entropy is largely undetermined: there is a family of entropies, all compatible with the Clausius-Duhem inequality, such that the difference between any two of them is not identically constant along processes of the body.

For this reason, Day's paper must be considered as a datum in the criticism of foundations of continuum thermodynamics.

In fact, as also scientists involved in practical problems in thermodynamics would agree, entropy does not enter the formulation of initial-boundary value problems; rather, it is used in connection with a dissipation inequality to deduce restrictions for the constitutive functionals.

The present paper points out that the indetermination of entropy plays a role also in global equivalence for bodies with fading memory.

Indeed, only the response functions for the 'elastic parts' of entropy of two bodies that are globally equivalent are mutually related, so that just in bodies with memory the 'dynamic' parts of entropy remain unrelated.

In this sense the results of the present paper are directly connected with Day's opinion about the role of entropy in continuum thermodynamics.

1.2 On the notion of global equivalence

The notion of global physical equivalence for continuous simple bodies has been introduced in [2] in the case of heat-conducting elastic bodies. Then, in [3], a weaker version of this notion has been considered in connection with simple bodies with fading memory.

The strong version [2] of the notion of global equivalence is understood in the following sense: two heat-conducting elastic bodies $\mathcal{B}$ and $\mathcal{B}'$ are globally physically equivalent if there exists a suitable bijective correspondence k between the bodies $\mathcal{B}$ and $\mathcal{B}'$ such that any global thermokinetic process for $\mathcal{B}$ is admissible (in the sense of Coleman and Noll) if and only if its corresponding by k for $\mathcal{B}'$ is also admissible. Here by global thermokinetic process we mean the solution of any given initial-boundary value problem.

Notice that an initial-boundary value problem for $\mathcal{B}$ is related with the assignment of two parts, s and z, of the boundary $\partial\mathcal{B}$ of $\mathcal{B}$, such that

(i) the traction boundary condition is prescribed on s,

(ii) the displacement boundary condition is prescribed on $\partial\mathcal{B} \setminus s$,

(iii) the heat flux boundary condition is prescribed on z,

(iv) the temperature boundary condition is prescribed on $\partial\mathcal{B} \setminus z$.

The strong version of global equivalence involves all thermokinetic processes, whose boundary conditions are prescribed, as in $(i) - (iv)$, on any arbitrarily given pair (s, z) on $\partial\mathcal{B}$.

Instead, the weak version of global equivalence involves only thermokinetic processes whose boundary conditions are prescribed on *one given pair* of boundary portions (s, z).

It turns out that *$\mathcal{B}$ and $\mathcal{B}'$ are strongly globally equivalent if and only if they are weakly globally equivalent with respect to each choice of (s, z).*

Moreover, in [3] the notion of entropy-equivalence is introduced for the response functionals $\hat{\mathbf{q}}$ and $\hat{\mathbf{q}}'$ of the heat flux in bodies $\mathcal{B}$ and $\mathcal{B}'$ that are globally equivalent: $\hat{\mathbf{q}}$ and $\hat{\mathbf{q}}'$ are said to be *entropy κ-equivalent* if they give the same contribution to the rate of increase of entropy due to heat conduction across any two k-corresponding surfaces. Hence, given a simple body $\mathcal{B}$, the class of all simple bodies that are globally equivalent to it is characterized both when the heat flux response functionals of these bodies are entropy-equivalent and when they are not.

These characterizations put in evidence the existence of globally equivalent bodies whose heat fluxes are not entropy-equivalent.

The theorems of global equivalence in [2]- [3] do not involve the entropy notion and any consequence of the Clausius-Duhem inequality; thus they can be interpreted within any thermodynamic theory where a dissipation principle is assumed. This is strictly related with the fact that the formulation of an initial-boundary value problem for a given body does not involve entropy.

1.3 About the present paper

In the present paper

- we state and prove a version of the above general theorem of global equivalence in a theory where the Clausius-Duhem inequality is assumed;

- we characterize the relation between the response functions of the elastic part of the entropy in any two simple bodies with fading memory that are globally equivalent; it turns out that the dynamic parts of entropy, η^d and η'^d are not constrained by the relation of global equivalence; this fact agrees with the indetermination of entropy emphasized by Day [1];

- furthermore, we show that the elastic parts of the heat fluxes in any two such bodies are entropy-equivalent.

2 Preliminaries

2.1 Thermodynamic processes for bodies with memory

We choose a reference configuration γ for the heat-conducting simple body $\mathcal{B}$ and assume that the region $\mathbf{B} = \gamma(\mathcal{B})$ of the three-dimensional ambient space is closed, bounded, simply connected and regular in Kellogg's sense. Hence $\mathbf{X} = \gamma(\mathcal{X}) \in \mathbf{B}$ is the place occupied by the material point $\mathcal{X} \in \mathcal{B}$ in the reference configuration.

A *thermokinetic process* for $\mathcal{B}$ for γ is here defined to be a pair

$$p = (\mathbf{x}(.), \theta(.)) \tag{1}$$

of smooth functions, where

$$\mathbf{x}(.) \ : \ \mathbf{B} \times \mathbf{R} \longrightarrow \mathbf{R}^3 \qquad \mathbf{x} = \mathbf{x}(\mathbf{X},\, t) \tag{2}$$

$$\theta(.) \ : \ \mathbf{B} \times \mathbf{R} \longrightarrow \mathbf{R}^+ \qquad \theta = \theta(\mathbf{X},\, t) \ , \tag{3}$$

such that $\mathbf{x} = \mathbf{x}(\mathbf{X},\, t)$ and $\theta = \theta(\mathbf{X},\, t)$ denote the position and temperature at $(\mathbf{X},\, t)$, respectively. Henceforth, we may also assume that the motion $\mathbf{x}(.)$ is an orientation preserving mapping, i.e.

$$\det \operatorname{Grad} \mathbf{x}(\mathbf{X},\, t) \ > \ 0 \qquad \forall\, (\mathbf{X},\, t) \in \mathbf{B} \times \mathbf{R}. \tag{4}$$

Let

$$\rho_0 = \hat{\rho}_0(\mathbf{X}) \ , \qquad \mathbf{X} \in \mathbf{B} \ , \tag{5}$$

denote the mass-density of $\mathcal{B}$ in the reference configuration.

We assume that at any $\mathbf{X} \in \mathbf{B}$ the first Piola-Kirchhoff stress tensor $\mathbf{P}$, heat flux vector $\mathbf{q}$, specific internal energy e and specific internal entropy η are determined by the total histories of the deformation gradient, temperature and material temperature gradient, respectively. Hence, there are functionals $\hat{\mathbf{P}}$, $\hat{\mathbf{q}}$, $\hat{e}$ and $\hat{\eta}$ such that

$$\mathbf{P} = \hat{\mathbf{P}}(\mathbf{F},\ \theta,\ \mathbf{G},\ \mathbf{F}^t,\ \theta^t,\ \mathbf{G}^t,\ \mathbf{X}), \tag{6}$$

$$\mathbf{q} = \hat{\mathbf{q}}(\mathbf{F},\ \theta,\ \mathbf{G},\ \mathbf{F}^t,\ \theta^t,\ \mathbf{G}^t,\ \mathbf{X}), \tag{7}$$

$$e = \hat{e}(\mathbf{F},\ \theta,\ \mathbf{G},\ \mathbf{F}^t,\ \theta^t,\ \mathbf{G}^t,\ \mathbf{X}), \tag{8}$$

$$\eta = \hat{\eta}(\mathbf{F},\ \theta,\ \mathbf{G},\ \mathbf{F}^t,\ \theta^t,\ \mathbf{G}^t,\ \mathbf{X}), \tag{9}$$

where

$$\mathbf{F} = \operatorname{Grad} \mathbf{x}(\mathbf{X},\ t) \ , \qquad \mathbf{G} = \operatorname{Grad} \theta(\mathbf{X},\ t) \tag{10}$$

are the deformation gradient and material temperature gradient, and

$$\mathbf{F}^t = \operatorname{Grad} \mathbf{x}(\mathbf{X},\ .) \ , \qquad \theta^t = \theta^t(\mathbf{X},\ .) \ , \qquad \mathbf{G}^t = \operatorname{Grad} \theta^t(\mathbf{X},\ .) \ , \tag{11}$$

are the past histories of $\mathbf{F}$, θ and $\mathbf{G}$, respectively, in the thermokinetic process (1).

A *thermodynamic process* in $\mathcal{B}$ is described by the following eight functions of $\mathbf{X}$ and t (see [4]):

$(p.1-2)$ The *motion* and *temperature* (2)-(3).

$(p.3)$ The first Piola-Kirchhoff *stress tensor* $\mathbf{P} = \mathbf{P}(\mathbf{X},\, t)$.

$(p.4)$ The specific *internal energy* $e = e(\mathbf{X},\, t)$, per unit mass.

(p.5) The specific *entropy* $\eta = \eta(\mathbf{X},\, t)$, per unit mass.

(p.6) The *heat flux* vector $\mathbf{q} = \mathbf{q}(\mathbf{X},\, t)$.

(p.7) The *body force* $\mathbf{b} = \mathbf{b}(\mathbf{X},\, t)$, per unit volume, exerted on $\mathcal{B}$ at $\mathbf{X}$ by the *external world.*

(p.8) The *heat supply* $r = r(\mathbf{X},\, t)$, which is the radiation energy, per unit mass and time, absorbed by $\mathcal{B}$ at $\mathbf{X}$ and furnished by the external world.

We say that the set of functions (p.1 − 8) is a *thermodynamic process* if it is compatible with the integral balance laws of *linear momentum*, *moment of momentum* and *energy.*

These integral laws, under suitable smoothness assumptions, are equivalent to the following local laws:

$$\rho_0 \ddot{\mathbf{x}} \;=\; \mathbf{b} + \mathrm{Div}\mathbf{P}\;, \tag{12}$$

$$\mathbf{P}\mathbf{F}^T \;=\; \mathbf{F}\mathbf{P}^T, \tag{13}$$

$$\rho_0 \dot{e} \;=\; \rho_0 r + \mathbf{P}\cdot\dot{\mathbf{F}} - \mathrm{Div}\mathbf{q}\;. \tag{14}$$

A thermodynamic process is determined by the six functions (p.1) − (p.6) above, the remaining two functions (p.7) − (p.8) being determined by the local balance laws (12) and (14).

A thermodynamic process is said to be *admissible* in $\mathcal{B}$ if it is compatible with the constitutive relations (6)-(9) at each $(\mathbf{X},\, t)$.

As it is well known, the assertion below holds.

1 Remark. *For every choice of the thermokinetic process* (1) *there exists a unique admissible thermodynamic process for $\mathcal{B}$.*

2.2 On the fading memory assumption

As it is customary, *Lin* denotes the vector space of (second-order) tensors on the vector space $\mathbf{R}^3$ and Lin^+ denotes the subset of *Lin* formed by the tensors with positive determinant.

We adopt the *fading memory assumption* such as in [5] and [4] although a more general way to directly link fading memory to the response may be deduced by generalizing [6].

Hence, we assume the existence of an *influence function*, describing the rate at which the memory fades; this is a monotone-decreasing, bounded, square integrable and positive function $h(s)$ of s, $\;0 \le s < \infty$.

For

$$\mathbf{\Lambda}(.) : (0, \infty) \to Lin^{+} \times \mathbf{R} \times \mathbf{R}^{3}\,, \qquad \mathbf{\Lambda}(.) \;=\; (\mathbf{F}^{t}(.),\; \theta^{t}(.),\; \mathbf{G}^{t}(.))\,, \tag{15}$$

let

$$|\mathbf{\Lambda}(s)|^{2} \;=\; \mathbf{\Lambda}(s)\cdot\mathbf{\Lambda}(s) \;=\; F^{t\,a}_{\;B}(s)\,F^{t\,a}_{\;B}(s) \;+\; \theta^{t\,2}(s) \;+\; G^{t}_{\;A}(s)\,G^{t}_{\;A}(s)\,, \tag{16}$$

where repetition of indexes stands for summation.

The collection $\mathcal{H}$ of all measurable functions (15) such that

$$\|\mathbf{\Lambda}(.)\|_{h} := [\int_{0}^{\infty} |\mathbf{\Lambda}(s)|^{2} h^{2}(s)\,ds\,]^{1/2} \;<\; \infty \tag{17}$$

is a Hilbert space with norm given by (17).

The *principle of fading memory* (see [4]) states that at each admissible $\mathbf{\Lambda} = (\mathbf{F},\ \theta,\ \mathbf{G})$ all the response functionals (6)-(9) have for their common domain $\mathcal{D}$ a neighborhood in $\mathcal{H}$ of the constant history $\mathbf{\Lambda}^{*}(s) \;=\; \mathbf{\Lambda} \quad \forall s > 0$, and they are Fréchet-differentiable throughout $\mathcal{D}$ with respect to the h-norm (17).

2.3 Initial-boundary-value problems

We consider thermokinetic processes in a time interval $Int = [t_0,\ t_1]$, with initial conditions at time t_0, that are controlled by an external action defined in $(-\infty,\ t_1]$. Of course, in practice the memory has a finite length d and the controlling action can be regarded as defined in the time interval $(t_o - d,\ t_1]$.

2 Definition. An *action up to the time* t_1 *of the external world* exerted on the heat-conducting simple body $\mathcal{B}$, referred to γ, is given by the prescriptions of

$(A.i)$ a body force $\mathbf{b} = \mathbf{b}(\mathbf{X},\ t)$, defined in $\gamma(\mathcal{B}) \times (t_o - d,\ t_1]$;

$(A.ii)$ a heat supply $r = r(\mathbf{X},\ t)$, defined in $\gamma(\mathcal{B}) \times (t_o - d,\ t_1]$;

$(A.iii)$ traction and displacement boundary conditions, that is, a contact force per unit area, $\mathbf{c} = \mathbf{c}_s(\mathbf{X},\ t)$, and a displacement, $\mathbf{x} = \mathbf{x}_{\overline{s}}(\mathbf{X},\ t)$, respectively defined on two complementary (Lebesgue) measurable parts s and $\overline{s}$ of the boundary $\partial\gamma(\mathcal{B}) \times (t_o - d,\ t_1]$, possibly with $s = \emptyset$ or $\overline{s} = \emptyset$;

$(A.iv)$ heat flux and temperature boundary conditions, that is, a normal heat flux $q = q_z(\mathbf{X},\ t)$ per unit area and time, and a temperature $\theta = \theta_{\overline{z}}(\mathbf{X},\ t)$, respectively defined on two complementary parts z and $\overline{z}$ of the boundary $\partial\gamma(\mathcal{B}) \times (t_o - d,\ t_1]$, possibly with $z = \emptyset$ or $\overline{z} = \emptyset$.

Briefly, we say that $(\mathbf{b},\ r,\ \mathbf{c}_s,\ \mathbf{x}_{\overline{s}},\ q_z,\ \theta_{\overline{z}})$ is an *external action* on $\mathcal{B}$, referred to γ, with boundary conditions prescribed on the parts s and z of $\partial\gamma(\mathcal{B})$.

3 Definition. The *initial-boundary-value-problem* for $\mathcal{B}$, subject to the external action

$$(\mathbf{b},\, r,\, \mathbf{c}_s,\, \mathbf{x}_{\overline{s}},\, q_z,\, \theta_{\overline{z}})\,,$$

consists in finding two (smooth) functions

$$\mathbf{x} = \mathbf{x}(\mathbf{X},\, t) \quad \textit{and} \quad \theta = \theta(\mathbf{X},\, t)$$

such that (i) the balance laws (12)-(14) hold, when the equalities (6)-(8), (10) and (11) are used, (ii) the boundary conditions $(A.iii) - (A.iv)$ hold and (iii) $\mathbf{x}(.,\, t)$, $\partial\mathbf{x}(.,\, t)/\partial t$ and $\theta(.,\, t)$ are given at $t = t_0$.

Note that, by Remark 1, corresponding to any thermokinetic process which solves an initial-boundary-value problem for $\mathcal{B}$ there exists a thermodynamic process in $\mathcal{B}$. These processes do not involve cuts, i.e. no space-like discontinuities arise.

2.4 Notions of global equivalence

Now, from [2] and [3], we rewrite the definitions of global equivalence.

Definition 4 below gives the notion of weak global equivalence, that is to say, global equivalence w.r.t. $(s,\, z)$ for **one** choice of the couple $(s,\, z)$ of parts on $\partial\mathcal{R}$; this notion defines *globally equivalent configurations,* γ *and* γ'*, for the simple bodies* $\mathcal{B}$ *and* $\mathcal{B}'$*, with respect to external actions whose boundary conditions are prescribed on the (Lebesgue) measurable parts* $s,\, \overline{s},\, z,\, \overline{z}$ *of the boundary of the region* $\mathcal{R} := \gamma[\mathcal{B}] = \gamma'[\mathcal{B}]$ for **one** choice of the couple $(s,\, z)$ of parts on $\partial\mathcal{R}$.

Definition 6 gives the notion of (strong) global equivalence, that is to say, global equivalence w.r.t. $(s,\, z)$ for **each** choice of the couple $(s,\, z)$ of parts on $\partial\mathcal{R}$;

One should identify these external actions with a suitable subclass of external actions connected to some $s,\, z \subset \partial\mathcal{R}$, for a suitable choice of $(s,\, z)$.

In order to state the aforementioned definitions we need the following assertion.

(2.A) *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be simple bodies. Let* γ *and* γ' *be (smooth) configurations, i.e.* C^2-*diffeomorphisms, for* $\mathcal{B}$ *and* $\mathcal{B}'$*, respectively, with* $\mathcal{R} := \gamma(\mathcal{B}) = \gamma'(\mathcal{B}')$*, where* $\mathcal{R}$ *is the closure of a bounded region that is regular in Kellogg's sense. Put* $k := \gamma'^{-1} \circ \gamma$ *and let*

$$(\hat{\rho}_0,\; \hat{\mathbf{P}},\; \hat{\mathbf{q}},\; \hat{e},\; \hat{\eta}) \tag{18}$$

and

$$(\; \hat{\rho}_0{}',\; \hat{\mathbf{P}}',\; \hat{\mathbf{q}}',\; \hat{e}',\; \hat{\eta}'\;) \tag{19}$$

be respective response functionals for $\mathcal{B}$ *and* $\mathcal{B}'$ *which have the forms (6) to (9) and have the same functional domain equipped with the same topology.*

4 Definition (**Weak global equivalence**).

(a) Assume (2.A). We say that the configurations γ and γ', respectively for the bodies $\mathcal{B}$ and $\mathcal{B}'$, are globally equivalent with respect to external actions with boundary conditions prescribed on the parts s, $\overline{s}$, z, $\overline{z}$ of $\partial\mathcal{R}$ $-$ briefly globally equivalent w.r.t. $(s,\, z)$, if

(i) $\hat{\rho} = \hat{\rho}'_0$ in $\mathcal{R}$;

(ii) under each given (smooth) external action $\mathcal{A} = (\mathbf{b},\, r,\, \mathbf{c}_s,\, \mathbf{x}_{\overline{s}},\, q_z,\, \theta_{\overline{z}})$, with boundary conditions prescribed on the parts s, $\overline{s}$, z, $\overline{z}$ of $\partial\gamma(\mathcal{B})$, in the time interval $(t_o - d,\ t_1]$, the functions $\mathbf{x} \circ \gamma$ and $\theta \circ \gamma$ in (2)-(3) represent (in γ) a thermokinetic process for $\mathcal{B}$ with the initial conditions

$$\mathbf{x}(\mathbf{X},\, t_0) = \mathbf{x}_0(\mathbf{X})\,, \quad \frac{\partial \mathbf{x}(\mathbf{X},\, t_0)}{\partial t} = \mathbf{v}_0(\mathbf{X})\,, \quad \theta(\mathbf{X},\, t_0) = \theta_0(\mathbf{X})\,,$$
$$(\mathbf{X} = \gamma(X), \quad X \in \mathcal{B}) \tag{20}$$

if and only if

under the action $\mathcal{A}$ in the time interval $(t_o - d,\ t_1]$ the functions $\mathbf{x} \circ \gamma'$ and $\theta \circ \gamma'$ in (1) represent (in γ') a thermokinetic process for $\mathcal{B}'$ with the initial conditions (20), where $\mathbf{X} = \gamma'(X'),\ X' \in \mathcal{B}'$.

(b) We say that the bodies $\mathcal{B}$ and $\mathcal{B}'$ are globally k-equivalent w.r.t. $(s,\, z)$, if they have two globally equivalent configurations w.r.t. $(s,\, z)$, γ and γ', such that $k = \gamma'^{-1} \circ \gamma$.

5 Remark. In the sequel, the processes $\big(\mathbf{x} \circ \gamma,\ \theta \circ \gamma\big)$ and $\big(\mathbf{x} \circ \gamma',\ \theta \circ \gamma'\big)$, mentioned in the above definition, will be called *k-corresponding processes*, with $k = \gamma'^{-1} \circ \gamma$.

6 Definition (**Strong global equivalence**).

(a) We say that γ and γ' are globally equivalent configurations for the bodies $\mathcal{B}$ and $\mathcal{B}'$ if γ and γ' are globally equivalent configurations w.r.t. $(s,\, z)$ for each choice $(s,\, z)$.

(b) We say that the bodies $\mathcal{B}$ and $\mathcal{B}'$ are globally k-equivalent if they have two globally equivalent configurations γ and γ' such that $k = \gamma'^{-1} \circ \gamma$.

7 Remark. Global equivalence w.r.t. $(s,\, z)$ and global equivalence are equivalence relations for the class of simple bodies with fading memory; hence, in particular, they are transitive.

3 Theorems of global equivalence with no reference to entropy

In the present section we rewrite the maximality theorems of weak and strong global equivalence written in [3]; these theorems characterize the relations between the response functionals $(\hat{\mathbf{P}},\ \hat{\mathbf{q}},\ \hat{e})$ and $(\hat{\mathbf{P}}',\ \hat{\mathbf{q}}',\ \hat{e}')$ of any two globally equivalent simple bodies.

As already noted in the introduction, here we shall prove a theorem of global equivalence involving entropy functionals, within a thermodynamic theory with the Clausius-Duhem inequality.

It characterizes the class of all simple bodies that are globally equivalent to a given such a body; hence it can be called *maximality theorem of global equivalence.*

3.1 Maximality theorem of weak global equivalence

For $\mathbf{F}^* \in Lin^+$, $\theta^* \in I\!R^+$ and $\mathbf{G}^* \in I\!R^3$ let $\mathbf{F}^{*t}$, θ^{*t} and $\mathbf{G}^{*t}$ denote the constant past histories whose respective values are $\mathbf{F}^*$, θ^* and $\mathbf{G}^*$.

For the sake of brevity we also put

$$\mathbf{\Lambda} = (\,\mathbf{F},\,\theta,\,\mathbf{G}\,)\,, \qquad \mathbf{\Lambda}^t = (\,\mathbf{F}^t,\,\theta^t,\,\mathbf{G}^t\,)\,. \tag{21}$$

The following statements of frame-indifference are used as hypotheses below.

(GI) *The response functionals of $\mathcal{B}$ and $\mathcal{B}'$ are Galilean invariant.*

(EI) *The response functionals of $\mathcal{B}$ and $\mathcal{B}'$ are Euclidean invariant.*

Note that the term *$k-$corresponding processes* appears frequently below; its meaning is explained in Definition 4 near equality (20).

8 Theorem. *Assume $(2.A)$, (GI) and let s and z be parts of $\partial\mathcal{R}$ with positive (Lebesgue) area. Then, the bodies $\mathcal{B}$ and $\mathcal{B}'$ are globally k-equivalent w.r.t. $(s,\,z)$*

if and only if

the assertions (a) and (b) below are true.

(a) *The response functionals for the stress and internal energy in $\mathcal{B}$ and $\mathcal{B}'$ are related by the equalities*

$$\hat{\mathbf{P}}'(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\mathbf{X}) = \hat{\mathbf{P}}(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\mathbf{X}) + Jp\mathbf{F}^{-T}\,, \tag{22}$$

$$\hat{e}'(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\mathbf{X}) = \hat{e}(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\,\mathbf{X}) + \rho_o^{-1}Jp + \hat{f}(\mathbf{\Lambda}^{*t},\,\mathbf{X})\,, \tag{23}$$

where $J = \det\mathbf{F}$ and

(i) *p is any constant for $s = \emptyset$,*

(ii) *$p = 0$ for $s \neq \emptyset$,*

(iii) *$\hat{f}$ is any smooth function of $\mathbf{\Lambda}^{*t}$, $\mathbf{X}$.*

(*b*) *The difference* $\hat{\mathcal{Q}} := \hat{\mathbf{q}}' - \hat{\mathbf{q}}$ *between the response functionals for the heat flux in* $\mathcal{B}$ *and* $\mathcal{B}'$,

$$\mathcal{Q} = \hat{\mathcal{Q}}(\mathbf{\Lambda}, \mathbf{\Lambda}^t, \mathbf{X}) = (\hat{\mathbf{q}}' - \hat{\mathbf{q}})(\mathbf{\Lambda}, \mathbf{\Lambda}^t, \mathbf{X}), \tag{24}$$

satisfies the equality

$$Div\mathcal{Q} = 0 \tag{25}$$

along any pair of $k-$*corresponding processes of* $\mathcal{B}$ *and* $\mathcal{B}'$*; moreover its components are given by*

$$\begin{aligned} \mathcal{Q}^A = {} & \overset{[1]}{\Gamma}{}^A + \varepsilon^{ABC} \overset{[2]}{\Gamma}_C \, G_B + \varepsilon^{ABC} < \overset{[3]}{\Gamma}_C \; | G^t_B > \\ & + \varepsilon^{ABC} < \overset{[4]}{\Gamma}_{ab} \; | F^{t\,b}_C > F^a_B + \varepsilon^{ABC} < \overset{[5]}{\Gamma} \; | G^t_B > G_C \end{aligned} \tag{26}$$

for any choice of

$$\overset{[4]}{\Gamma}_{ab} \in \mathbf{R} \tag{27}$$

and for any choice of the smooth functions

$$\overset{[1]}{\Gamma}{}^A, \; \overset{[2]}{\Gamma}_C, \; \overset{[3]}{\Gamma}_C, \; \overset{[5]}{\Gamma} \tag{28}$$

of θ, θ^t, $\mathbf{X}$, *such that*

$$\frac{\partial \overset{[1]}{\Gamma}{}^A}{\partial X^A} = 0, \qquad \text{i.e.} \qquad \overset{[1]}{\mathbf{\Gamma}} = Curl \overset{[1]}{\mathbf{w}} \quad \text{with} \quad \overset{[1]}{\mathbf{w}} = \overset{[1]}{\mathbf{w}} (\theta, \theta^t, \mathbf{X}), \tag{29}$$

$$\frac{\partial \overset{[1]}{\Gamma}{}^D}{\partial \theta} + \varepsilon^{ADC} \frac{\partial \overset{[2]}{\Gamma}_C}{\partial X^A} = 0, \qquad \text{i.e.} \qquad \frac{\partial \overset{[1]}{\mathbf{\Gamma}}}{\partial \theta} = Curl \overset{[2]}{\mathbf{\Gamma}}, \tag{30}$$

$$\frac{\partial \overset{[1]}{\Gamma}{}^D}{\partial \theta^t} + \varepsilon^{ADC} \frac{\partial \overset{[3]}{\Gamma}_C}{\partial X^A} = 0, \qquad \text{i.e.} \qquad \frac{\partial \overset{[1]}{\mathbf{\Gamma}}}{\partial \theta^t} = Curl \overset{[3]}{\mathbf{\Gamma}}, \tag{31}$$

and

$$\frac{\partial \overset{[3]}{\Gamma}_C}{\partial \theta} - \frac{\partial \overset{[2]}{\Gamma}_C}{\partial \theta^t} - \frac{\partial \overset{[5]}{\Gamma}}{\partial X^C} = 0. \tag{32}$$

In case $z \neq \emptyset$, *we also have*

$$\overset{[4]}{\Gamma}_{ab} = 0, \tag{33}$$

and, at any point of z,

$$\overset{[5]}{\Gamma} = 0, \qquad \overset{[1]}{\mathbf{\Gamma}} \cdot \mathbf{N} = 0, \quad \overset{[i]}{\mathbf{\Gamma}} \times \mathbf{N} = \mathbf{0} \quad \text{for } i = 2, 3, \tag{34}$$

where $\mathbf{N}$ *is the outward unit normal to* z.

If, in addition, $\mathcal{Q}$ *is Euclidean invariant, then* $\overset{[4]}{\Gamma}_{ab} = 0$.

3.2 Maximality theorem of strong global equivalence

9 Theorem. *Assume* (2.*A*) *and* (*GI*)*. Then the bodies* $\mathcal{B}$ *and* $\mathcal{B}'$ *are globally* k*-equivalent*

if and only if

(a) $\hat{\mathbf{P}}' = \hat{\mathbf{P}}$,
(b) *there exists a function* $f = \hat{f}(\mathbf{F}^{*t},\ \theta^{*t}, \mathbf{G}^{*t},\ \mathbf{X})$ *such that*

$$\hat{e}'(\mathbf{F},\theta,\mathbf{G},\mathbf{F}^t,\theta^t,\mathbf{G}^t,\mathbf{X}) = \hat{e}(\mathbf{F},\theta,\mathbf{G},\mathbf{F}^t,\theta^t,\mathbf{G}^t,\mathbf{X}) + \hat{f}(\mathbf{F}^{*t},\theta^{*t},\mathbf{G}^{*t},\mathbf{X}), \quad (35)$$

and

(c) $\hat{\mathbf{q}}'$ *and* $\hat{\mathbf{q}}$ *are such that the assertion* (b) *of Theorem 8 above is true from (26) to (32), the equality (33) holds and the equalities (34) hold at each point of the boundary* $\partial\mathcal{R}$*.*

4 On global equivalence in a theory with the Clausius-Duhem inequality

In the first two subsections of the present section we remind:
(i) the restrictions imposed by the Clausius-Duhem inequality on the response functionals of a simple body with memory, $\mathcal{B}$;
(ii) the definition of *elastic response function* associated with any response functional of $\mathcal{B}$.

Hence, in the third subsection we prove Theorem 11, which yields the relation between the elastic entropies of any two bodies which are weakly globally equivalent. This relation can be considered as the indetermination in the response function of elastic entropy in a given $\mathcal{B}$.

The last theorem, jointly with Theorem 8, allow for proving here the maximality theorem of weak global equivalence, i.e.Theorem 12. The latter admits the strong global equivalence version, i.e. Theorem 13.

4.1 Restrictions on the response functionals imposed by the Clausius-Duhem inequality

As is well known, the Clausius-Duhem inequality is postulated to hold along any admissible thermodynamic process of the body (see [4], p.15). Its local form is

$$\rho\,\dot{\eta} \,\geq\, \rho\theta^{-1}r \,-\, \mathrm{Div}(\theta^{-1}\mathbf{q})\,. \quad (36)$$

This inequality is equivalent to the *reduced dissipation inequality*

$$\rho(\dot{\psi} + \dot{\theta}\eta) \,-\, \mathbf{P}\cdot\dot{\mathbf{F}} \,+\, \theta^{-1}\mathbf{q}\cdot\mathbf{G} \,\leq\, 0\,, \quad (37)$$

where

$$\psi \,=\, e \,-\, \theta\eta \tag{38}$$

is the *specific free-energy.*

The validity of the Clausius-Duhem inequality along any process, exploited through the Coleman and Noll's procedure, implies the well known restrictions on the response functionals of the body

$$\mathbf{P} \,=\, \frac{\partial\psi}{\partial\mathbf{F}}\,, \quad \eta \,=\, -\frac{\partial\psi}{\partial\theta}\,, \quad \frac{\partial\psi}{\partial\mathbf{G}} \,=\, \mathbf{O}\,, \quad \delta\psi \,+\, (\rho\theta)^{-1}\mathbf{q}\cdot\mathbf{G} \,\le\, 0\,. \tag{39}$$

4.2 Elastic response functions and Gibbs relation

Consider a global thermokinetic process (1)-(3); we say that the triple

$$\mathbf{\Lambda} = \mathbf{\Lambda}(t) = (\mathbf{F}(t),\, \theta(t),\, \mathbf{G}(t))\,,$$

induced by (1)-(3) at a given $\mathbf{X} \in \gamma(\mathcal{B})$, is a *local process* at $\mathbf{X}$.

Now, in analogy with [7] on p.98, let $q(.) \in C^2(I\!R,\, I\!R)$ be a monotone increasing function such that

$$q(\zeta) = 0 \quad \text{for} \quad \zeta \le 0 \qquad \text{and} \qquad q(\zeta) = 1 \quad \text{for} \quad \zeta \ge 1\,;$$

hence for $\mathbf{\Lambda},\, \mathbf{\Lambda}^* \in Lin^+ \times I\!R^+ \times I\!R^3$ and $\varepsilon > 0$ put

$$\mathbf{\Lambda}_\varepsilon(\tau) \,=\, \mathbf{\Lambda}^*\, 1(\tau) \,+\, q(\frac{\tau - t + \varepsilon}{\varepsilon})\,(\mathbf{\Lambda} - \mathbf{\Lambda}^*)\,, \tag{40}$$

with $1(\tau)$ denoting the constant history whose value is 1 for each $\tau \in R$.

Then, in the limit as $\varepsilon \to 0$ the history $\mathbf{\Lambda}^t_\varepsilon(s)$, of $\mathbf{\Lambda}_\varepsilon(\tau)$ in (40) up to time $\tau = t$, tends to the jump-history $\mathbf{\Lambda}^t(.)$ corresponding to a strain-temperature impulse $\mathbf{\Lambda} - \mathbf{\Lambda}^*$ superimposed at time t to the rest history $\mathbf{\Lambda}^*(s)$, $\mathbf{\Lambda}^*(s) = \mathbf{O}$ for each $s \ge 0$. Such jump-histories correspond to Coleman's *fast processes* [8]. Of course, in the case $\mathbf{\Lambda} = \mathbf{\Lambda}^*$ it reduces to a history with constant continuation at time t.

Now, let $\mathbf{\Lambda}^t(.)$ be the history up to time t of a given local process $\mathbf{\Lambda}(\tau)$ at a fixed $\mathbf{X} \in \gamma(\mathcal{B})$, put

$$\mathbf{\Lambda} = \mathbf{\Lambda}(t)\,, \qquad \mathbf{\Lambda}^* = lim_{\varepsilon\to 0+}\mathbf{\Lambda}(t - \varepsilon)\,, \tag{41}$$

and let $\mathbf{\Lambda}^{*t}$ be the constant past history whose value is $\mathbf{\Lambda}^*$. Then the *elastic response functions associated with the response functionals of the body,* $\hat{\xi} \in \{\hat{\mathbf{P}},\, \hat{\mathbf{q}},\, \hat{e},\, \hat{\eta}\}$, are defined by

$$\xi^* \,=\, \hat{\xi}^*(\mathbf{\Lambda},\, \mathbf{\Lambda}^{*t},\, \mathbf{X})\,. \tag{42}$$

These correspond to the *pure-jump response functions* determined by the response functionals of the body, considered in [9]; of course, for $\mathbf{\Lambda} = \mathbf{\Lambda}^*$ they reduce to the usual *equilibrium response functions* determined by the response functionals of the body.

The Clausius-Duhem inequality implies the following restrictions

$$\mathbf{P}^* = \frac{\partial \psi^*}{\partial \mathbf{F}}\,, \quad \eta^* = -\frac{\partial \psi^*}{\partial \theta}\,, \quad \frac{\partial \psi^*}{\partial \mathbf{G}} = \mathbf{O}\,, \quad \delta\psi + (\rho\theta)^{-1}\mathbf{q}^* \cdot \mathbf{G} \leq 0\,. \tag{43}$$

for the elastic response functions (see, e.g., the proofs in [7], Section 5.4, for the equilibrium functions and in [9] for the pure-jump functions). These restrictions also imply

$$\frac{\partial \mathbf{P}^*}{\partial \mathbf{G}} = \mathbf{O}\,, \qquad \frac{\partial e^*}{\partial \mathbf{G}} = \mathbf{O}\,, \qquad \frac{\partial \eta^*}{\partial \mathbf{G}} = \mathbf{O}\,. \tag{44}$$

One also proves that (see [9] for the pure-jump functions, and e.g. [7] for the equilibrium functions) the Gibbs relation

$$\delta^* = \theta\,\dot{\eta}^* - \dot{e}^* + \rho_0^{-1}\,\mathbf{P}^* \cdot \dot{\mathbf{F}} = 0 \tag{45}$$

holds along any local process, where δ^* is regarded as the internal dissipation evaluated by using the elastic response functions.

The inequality $(43)_4$ implies the *piezo-caloric equality*

$$\mathbf{q}^*(\mathbf{F},\, \theta,\, \mathbf{0},\, \mathbf{\Lambda}^{*t},\, \mathbf{X}) = \mathbf{0}\,. \tag{46}$$

Lastly, the equality (25) implies that

$$\mathrm{Div}\mathbf{Q}^* = \mathbf{0} \tag{47}$$

along any pair of k-corresponding processes of $\mathcal{B}$ and $\mathcal{B}'$ (for the proof see [9]).

4.3 Theorem of global equivalence in a theory with the Clausius-Duhem inequality

The lemma below is used to prove the theorems of global equivalence in the thermodynamic theory with the Clausius-Duhem inequality, which are written below.

10 Lemma. *Let*

(i) *V be a Banach space with inner product;*

(ii) *$h(s)$ be an influence function;*

(iii) *$\mathcal{H}$ be the Hilbert space formed by the (smooth) histories*

$$x(s)\,:\,(0,\;\infty)\to V\,,\qquad x=x(s)\,,$$

with inner product

$$<x(s),\,y(s)>=\int_0^\infty x(s)\cdot y(s)\,h^2(s)\,ds\,,\qquad x(.),\,y(.)\,\in\,\mathcal{H}\,,\qquad (48)$$

where in $x(.)$, $y(.)$ the inner product of V is used.

(iv) $f=f\big(x(.)\big)\,\in\,C^1\big(\mathcal{S}\big)$ *with $\mathcal{S}$ subset of $\mathcal{H}$ which contains*

$$\mathcal{C}\,=\,\{\,c(.)\in\mathcal{H}\;:\;c(.)\text{ is constant in }(0,\,\infty)\,\}\;.$$

Then

$\dot f\,=\,0$ *for each* $x(.)\in\mathcal{S}$ *if and only if* $f=f\big(c(.)\big)$ *(i.e.,* $\mathcal{S}=\mathcal{C}$*).*

Proof. For $x(.)\in\mathcal{H}$ let $c_x(.)\in\mathcal{C}$ be such that

$$c_x(s):=x(0^+)\;\forall s\in(0,\,\infty),\qquad x(0^+):=lim_{\varepsilon\to0+}x(\varepsilon)\,,$$

and put

$$x_o(.):=x(.)-c_x(.),\qquad g=g\big(x_o(.),\,c_x(.)\big):=f\big(x_o(.)+c_x(.)\big)=f\big(x(.)\big)\,.$$

Note that $\frac{\partial g}{\partial c}\cdot\dot c_x(.)=0$ because $\|\dot c_x(.)\|_h=0$; hence we have

$$\dot g\,=\,\dot f\quad\text{and}\quad\dot g\,=\,\frac{\partial g}{\partial x_o}\dot x_0\,.$$

Thus $\dot f\,=0\;\forall x(.)\in\mathcal{S}$ if and only if $\frac{\partial g}{\partial x_o}\dot x_0\;=\;0$; by the arbitrariness of $x_0(.)$ the last equality is equivalent to $\frac{\partial g}{\partial x_o}\;=0$; hence $\dot f\,=0$ if and only if $g=g(c(.))$, *i.e.,* $f=f(c(.))$, $c(.)\in\mathcal{C}$. $\boxed{QED}$

11 Theorem. *Assume (2.A), (G.I) and let s, z be (Lebesgue) measurable portions of $\partial\mathcal{R}$.*

If the bodies $\mathcal{B}$ and $\mathcal{B}'$ are globally k-equivalent w.r.t. $(s,\,z)$, then there is a function $N^*=\hat N^*(\mathbf{\Lambda}^{*t},\,\mathbf{X})$ such that the response functions for the specific elastic entropy in $\mathcal{B}$ and $\mathcal{B}'$ are related by the equality

$$\hat\eta'^*(\mathbf{F},\,\theta,\,\mathbf{\Lambda}^{*t},\,\mathbf{X})\;=\;\hat\eta^*(\mathbf{F},\,\theta,\,\mathbf{\Lambda}^{*t},\,\mathbf{X})\;+\;\hat N^*(\mathbf{\Lambda}^{*t},\,\mathbf{X})\,.\qquad(49)$$

Proof. The Gibbs relation (45) holds along any pair of $k-$corresponding processes of $\mathcal{B}$ and $\mathcal{B}'$. Now, by subtracting (45) written for $\mathcal{B}'$'s functionals with the same written for $\mathcal{B}$'s functionals we find that the equality

$$\theta\,\dot{N}^* \,-\, \dot{E}^* \,+\, \rho_0^{-1}\left(\dot{\mathbf{P}}'^* - \dot{\mathbf{P}}^*\right)\cdot\dot{\mathbf{F}} \,=\, 0 \tag{50}$$

holds along any pair of $k-$corresponding processes of $\mathcal{B}$ and $\mathcal{B}'$, where

$$\hat{N}^* \,=\, \hat{\eta}'^* \,-\, \hat{\eta}^*\,, \qquad \hat{E}^* \,=\, \hat{e}'^* \,-\, \hat{e}^*\,. \tag{51}$$

Now the equalities (22) and (23) respectively yield

$$\left(\hat{\mathbf{P}}'(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\mathbf{X}) \,-\, \hat{\mathbf{P}}(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\mathbf{X})\right)\cdot\dot{\mathbf{F}} \,=\, Jp\mathbf{F}^{-T}\cdot\dot{\mathbf{F}}\,, \tag{52}$$

where $J = \det\mathbf{F}$, and

$$\hat{E} \,:=\, \hat{e}'(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\,\mathbf{X}) \,-\, \hat{e}(\mathbf{\Lambda},\,\mathbf{\Lambda}^t,\,\mathbf{X}) \,=\, \rho_o^{-1}Jp \,+\, \hat{f}(\mathbf{\Lambda}^{*t},\mathbf{X})\,, \tag{53}$$

with $J = \det\mathbf{F}$, and where

(i) $\hat{f}$ is any smooth function,

(ii) p is any constant when $s = \emptyset$,

(iii) $p = 0$ when $s \neq \emptyset$.

Thus by $(53)_2$ we have $E = \hat{E}(\mathbf{\Lambda}^{*t},\,\mathbf{X})$.
Now the Piola identity $\partial J/\partial\mathbf{F} = J\mathbf{F}^{-T}$ and (53) yield

$$\dot{E}^* \,=\, \rho_0^{-1}\left(\dot{\mathbf{P}}'^* - \dot{\mathbf{P}}^*\right)\cdot\dot{\mathbf{F}}\,. \tag{54}$$

This equality together with (50) yield

$$\dot{N}^* \,=\, 0 \tag{55}$$

along any process. Hence by Lemma 10 we have

$$N^* \,=\, \hat{N}^*(\mathbf{\Lambda}^{*t},\,\mathbf{X})\,. \tag{56}$$

$\boxed{QED}$

12 Theorem. *Assume* (2.A)*,* (G.I) *and let* s*,* z *be (Lebesgue) measurable portions of* $\partial\mathcal{R}$*. Then*

(α) *the bodies* $\mathcal{B}$ *and* $\mathcal{B}'$ *are globally* k*-equivalent w.r.t.* $(s,\,z)$*,* *if and only if*

(β) *the assertions* (a) *and* (b) *in Theorem 8 are verified and there is a function* $N^* = \hat{N}^*(\mathbf{\Lambda}^{*t}, \mathbf{X})$ *such that the response functions for the specific elastic entropy in* $\mathcal{B}$ *and* $\mathcal{B}'$ *are related by the equality (49).*

PROOF. Assume (α); then Theorems 8 and 11 yield (β). Conversely, assume (β); then Theorem 8 implies the global k-equivalence of $\mathcal{B}$ and $\mathcal{B}'$ w.r.t. (s, z). $\boxed{QED}$

In connection with (strong) global equivalence rather than global k-equivalence w.r.t. (s, z) it is easy to prove the version below of the last theorem. Its proof is based on Theorem 9 instead of Theorem 8.

13 Theorem. *Assume* (2.A) *and* ($G.I$). *Then*
(α) *the bodies* $\mathcal{B}$ *and* $\mathcal{B}'$ *are globally* k*-equivalent*

if and only if

(β) *the assertions* (a) *through* (c) *in Theorem 9 are verified and there is a function* $N^* = \hat{N}^*(\mathbf{\Lambda}^{*t}, \mathbf{X})$ *such that the response functions for the specific elastic entropy in* $\mathcal{B}$ *and* $\mathcal{B}'$ *are related by the equality (49).*

5 Entropy-equivalence of heat fluxes

In the present section (i) we recall from [3] the notion of entropy-equivalence for the heat-flux functionals of globally equivalent bodies; (i) we remind that there are simple bodies which are globally equivalent but such that their heat fluxes are not entropy-equivalent; in fact Theorem 6.1 of [3], reminded here in the third subsection, characterizes the class of simple bodies which are weakly globally equivalent but not entropy-equivalent to a given such a body.

Hence we show that the elastic parts of the heat flux in globally equivalent bodies are necessarily entropy-equivalent. As reminded in (ii) above this is not true for the entire heat fluxes.

5.1 Definition of entropy-equivalence for the response functionals of heat flux in globally equivalent bodies

The *contribution to the rate of increase of entropy, in any given part* $\mathcal{P}$ *of the simple body* $\mathcal{B}$*, due to heat conduction across the boundary* $\partial\mathcal{P}$ is

$$-\oint_{\partial\mathcal{P}} \frac{1}{\theta}\,\mathbf{q}\cdot\mathbf{N}\,dA\,,$$

see [10] on pages 118 – 119; see also the textbook [4] on pages 78 – 80. By the divergence theorem the volume density of this contribution is

$$-\mathrm{Div}(\frac{1}{\theta}\,\mathbf{q})\,.$$

Hence (from [2]) we say that the response functionals for the heat flux of two globally equivalent bodies are *entropy-equivalent*, if they have the same volume density of entropy production along every corresponding thermokinetic processes.

14 Definition. *Assume* (2.*A*) *and let* γ *and* γ' *be globally equivalent respective configurations for* $\mathcal{B}$ *and* $\mathcal{B}'$.

(a) We say that the heat flux response functionals $\hat{\mathbf{q}}$ *and* $\hat{\mathbf{q}}'$ *for* $\mathcal{B}$ *and* $\mathcal{B}'$ *are* entropy κ-equivalent, *if*

$$Div(\frac{1}{\theta}\,\mathbf{q}) \;=\; Div(\frac{1}{\theta}\,\mathbf{q}') \tag{57}$$

along any two $k-$*corresponding processes of* $\mathcal{B}$ *and* $\mathcal{B}'$.

(b) We say that the configurations γ *of* $\mathcal{B}$ *and* γ' *of* $\mathcal{B}'$ *are* entropy κ-equivalent, *with* $k=\gamma'^{-1}\circ\gamma$, *if* $\hat{\mathbf{q}}$ *and* $\hat{\mathbf{q}}'$ *are entropy* κ*-equivalent.*

(c) We say that $\mathcal{B}$ *and* $\mathcal{B}'$ *are* entropy equivalent, *or* entropy k-equivalent, *if they have two respective configurations* γ *and* γ' *that are entropy* $k-$*equivalent, with* $k=\gamma'^{-1}\circ\gamma$.

5.2 On global equivalence without entropy-equivalence

Let $\mathcal{B}$ be a simple body. The theorem below characterizes the form of the heat flux response functional for any simple body $\mathcal{B}'$ that is globally equivalent but not entropy-equivalent to $\mathcal{B}$. Incidentally, it shows the existence of globally equivalent bodies whose heat fluxes are not entropy-equivalent.

15 Theorem. *Assume* (2.*A*), (*G.I*) *and let* s, z *be (Lebesgue) measurable portions of* $\partial\mathcal{R}$. *Let* $\mathcal{B}$ *and* $\mathcal{B}'$ *be globally* k*-equivalent w.r.t.* $(s,\,z)$. *Then,*

the heat flux response functionals $\hat{\mathbf{q}}$ *and* $\hat{\mathbf{q}}'$ *are not entropy-equivalent*

if and only if

the difference $\hat{\mathcal{Q}} := \hat{\mathbf{q}}' - \hat{\mathbf{q}}$ *satisfies the conditions* (a) *and* (b) *below, where at least one of the quantities* $\overset{[1]}{\Gamma}{}^{A}$, $\overset{[3]}{\Gamma}{}_{C}$, $\overset{[4]}{\Gamma}{}_{ab}$ *does not vanish.*

The proof follows from Theorems 4.1 and 4.2 in [3].

5.3 Entropy-equivalence of the elastic heat fluxes in globally equivalent bodies

In the present section we show that the elastic parts of the heat flux in globally equivalent bodies always are entropy-equivalent.

16 Theorem. *Assume* (2.A), (GI) *and let* z *be any (Lebesgue) measurable part of* $\partial\mathcal{R}$.

Let the bodies $\mathcal{B}$ *and* $\mathcal{B}'$ *be globally* $k-$*equivalent for external actions whose heat boundary conditions is prescribed on* z.

Then the elastic response functions $\hat{\mathbf{q}}$ *and* $\hat{\mathbf{q}}'$ *for the heat flux, respectively in* $\mathcal{B}$ *and* $\mathcal{B}'$, *are entropy-equivalent,* i.e.,

$$Div(\frac{1}{\theta}\,\mathbf{q}^*) \;=\; Div(\frac{1}{\theta}\,\mathbf{q}'^*) \tag{58}$$

along any two $k-$*corresponding processes of* $\mathcal{B}$ *and* $\mathcal{B}'$.

Proof. Under the hypotheses of the theorem, thesis (b) of Theorem 8 holds. Hence by the equalities (26) and (42) we have
that the difference $\mathcal{Q}^* = \hat{\mathbf{q}}'^* - \hat{\mathbf{q}}^*$ has components

$$\begin{aligned} \mathcal{Q}^{*A} \;=\; \overset{[1]}{\Gamma}{}^{A} \;+\varepsilon^{ABC}\,\overset{[2]}{\Gamma}_C\,G_B \,+\, \varepsilon^{ABC} <\overset{[3]}{\Gamma}_C\;|\,G^{*t}_B> \\ +\,\varepsilon^{ABC}<\overset{[4]}{\Gamma}_{ab}\;|\,F^{*t\,b}_C> F^a_B \,+\, \varepsilon^{ABC}<\overset{[5]}{\Gamma}\;\;|\,G^{*t}_B> G_C \end{aligned} \tag{59}$$

By the piezo-caloric equality (46) the equality (59) yields

$$\overset{[1]}{\Gamma}{}^{A} \;+\varepsilon^{ABC}<\overset{[3]}{\Gamma}_C\;|\,G^{*t}_B> \;+\varepsilon^{ABC}<\overset{[4]}{\Gamma}_{ab}\;|\,F^{*t\,b}_C> F^a_B \;=\; 0$$

along any two $k-$corresponding processes of $\mathcal{B}$ and $\mathcal{B}'$; hence (59) becomes

$$\mathcal{Q}^{*A} \;=\; \varepsilon^{ABC}\,\overset{[2]}{\Gamma}_C\,G_B \,+\, \varepsilon^{ABC}<\overset{[5]}{\Gamma}\;\;|\,G^{*t}_B> G_C \tag{60}$$

and thus

$$\mathcal{Q}^*\cdot\mathbf{G} \;=\; \mathbf{0}\,. \tag{61}$$

Now the identity

$$\mathrm{Div}\Big(\theta^{-1}\,\mathcal{Q}^*\Big) \;=\; \theta^{-1}\mathrm{Div}\mathcal{Q}^* \;-\; \theta^{-2}\mathcal{Q}^*\cdot\mathbf{G}\,, \tag{62}$$

together with $\mathrm{Div}\mathcal{Q}^* \;=\; 0$, yield

$$\mathrm{Div}\Big(\theta^{-1}\,\mathcal{Q}^*\Big) \;=\; 0\,.$$

$\boxed{QED}$

References

[1] W. A. Day: *An objection to using entropy as a primitive concept in continuum thermodynamics*, Acta Mechanica, **27**, (1977), 251–255.

[2] A. Montanaro: *Global equivalence for deformable thermoelastic bodies*, Archive for Rational Mechanics and Analysis, **143**, (1998), 375–400.

[3] A. Montanaro: *Global Equivalence for Heat-Conducting Bodies with Fading Memory*, Mathematics and Mechanics of Solids, *to appear*.

[4] B. D. Coleman: *Thermodynamics of materials with memory*, Archive for Rational Mechanics and Analysis, **17**, (1964), 1–46.

[5] B. D. Coleman and W. Noll: *An approximation theorem for functionals, with applications in continuum mechanics*, Archive for Rational Mechanics and Analysis, **6**, (1960), 355–370.

[6] G. Del Piero and L. Deseri: *On the concept of state and free energy in linear viscoelasticity*, Archive for Rational Mechanics and Analysis, **38**, (1993), 197–236.

[7] W. A. Day: The Thermodynamics of Simple Materials with Fading Memory, Springer-Verlag, Berlin-Heidelberg-New York, (1972), 1–134.

[8] B. D. Coleman: *On Thermodynamics, Strain Impulses, ana Viscoelasticity*, Archive for Rational Mechanics and Analysis, **17**, (1964), 230–254.

[9] A. Montanaro, *Constitutive equations for quasi-processes of local pure-jump in simple materials with fading memory*, Atti Acc. Lincei, Mem. Mat. e Appl., s.9, **1**, (1993), 197–236.

[10] C. Truesdell: Rational Thermodynamics, Second Edition, Springer-Verlag, Berlin-Heidelberg-New York, 1984.